Enrollment No: ____________________ Exam Seat No: ____________________

C.U.SHAH UNIVERSITY

Winter Examination-2018

Subject Name: Linear Algebra

Subject Code: 5SC01LIA1 **Branch : M.Sc. (Mathematics)**

Semester: 1 **Date: 26/11/2018** **Time :02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 Attempt the Following questions (07)

a) If $v_1, v_2, \dots \dots \dots . v_n$ are in V then either they are linearly independent or some v_k is a linear combination of preceding one's $v_1, v_2, \dots \dots \dots . v_{k-1}$. (02)

b) Let V be finite dimensional over F and $T \in A(V)$ show that the number of characteristic root of T is atmost n^2. (02)

c) If A and B are finite dimensional subspaces of a vector space V, then $A + B$ is finite dimensional and $\dim(A + B) = \dim A + \dim B - \dim(A \cap B)$. (02)

d) Define: Minimal Polynomial for T. (01)

Q-2 Attempt all questions (14)

a) Let V and W be vector space over F of dimension m and n respectively. Then prove that $HOM(V, W)$ is of dimension mn over F. (07)

b) Let V be a finite dimensional vector space over $\boldsymbol{F}$, and W be a subspace of V then $\widehat{W}$ is isomorphic to $\hat{V}|W^{\circ}$ and $\dim W^{\circ} = \dim V - \dim W$. (07)

OR

Q-2 Attempt all questions (14)

a) Let V be a finite dimensional vector space over F and W be subspace of V. Show that W is finite dimensional ,$\dim W \leq \dim V$ and $\dim(V/W) = \dim V - \dim W$. (06)

b) State and prove Gram-Schmidth Orthonormalization process. (05)

c) If V is a finite dimensional inner product space and W is subspace of V then show that $W = (W^{\perp})^{\perp}$ (03)

Q-3 Attempt all questions (14)

a) Let V be a finite dimensional vector space over F and $S, T \in A(V)$.show that (06)

i) $rank(ST) \le rank(T)$

ii) $rank(TS) \le rank(T)$

iii) If S is regular then $rank(ST) = rank(TS) = rank(T)$

b) Let V be finite dimensional over F and $T \in A(V)$. If $\lambda_1, \lambda_2, \dots \dots \dots . \lambda_k$ in F are distinct roots of T and $v_1, v_2, \dots \dots \dots . v_k$ are characteristic vector of T corresponding to $\lambda_1, \lambda_2, \dots \dots \dots . \lambda_k$ respectively. Then $v_1, v_2, \dots \dots \dots . v_k$ are linearly independent. (05)

c) Let V be a finite dimensional vector space over F. If $T \in A(V)$ is right invertible then T is invertible. (03)

OR

Q-3 Attempt all questions (14)

a) If $\mathcal{A}$ is an algebra over F with unit element then prove that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. (05)

b) If V is finite dimensional over F, then prove that $T \in A(V)$ is invertible if and only if the constant term in the minimal polynomial for T is nonzero. (05)

c) Prove that $S \in A(V)$ is regular if and only if whenever $v_1, v_2, \dots \dots \dots . v_n \in V$ are linearly independent then $S(v_1), S(v_2), \dots \dots \dots . S(v_n)$ are also linearly independent. (04)

SECTION – II

Q-4 Attempt the Following questions (07)

a) Let F be a field of characteristic 0 and V be a vector space over F. If $S, T \in A(V)$ such that $ST - TS$ commutes with S then show that $ST - TS$ is nilpotent. (02)

b) Prove that there do not exists $A, B \in M_n(F)$ such that $AB - BA = I$, where F is field with characteristic 0. (02)

c) Find the inertia of quadratic equation $2x_1x_2 + 2x_1x_3 = 0$. (02)

d) Define: Invariant (01)

Q-5 Attempt all questions (14)

a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent then the invariants of T are unique. (07)

b) Let V be a finite dimensional vector space over F and $T \in A(V)$ and W be subspace of V invariant under T. Then T induce a map $\bar{T}: V/W \to V/W$ defined by $\bar{T}(v + W) = Tv + W$ show that $\bar{T} \in A(V/W)$. Further $\bar{T}$ satisfies every polynomial satisfies by T. If $p_1(x)$ and $p(x)$ are minimal polynomial for $\bar{T}$ and T respectively then show that $p_1(x)/p(x)$. (07)

OR

Q-5 Attempt all questions (14)

a) Let V be a finite dimensional vector space over F and $T \in A(V)$. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_1 = T\big|_{V_1}$ and $T_2 = T\big|_{V_2}$. If the minimal polynomial of T_1 over F is $p_1(x)$ while minimal polynomial of T_2 over F is $p_2(x)$. Then show that minimal polynomial (05) of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

b) Two nilpotent linear transformations are similar if and only if they have the same invariants. (05)

c) Find the invariants of linear transformation $T: R^3 \to R^3$ defined by $T(x, y, z) = (y, 0,0)$, where $x, y, z \in R$. (04)

Q-6 **Attempt all questions** **(14)**

a) Let $f: R^n \times R^n \to R$ be a map. Then f is bilinear if and only if there exist $\alpha_{ij} \in R$, $1 \le i, j \le n$ with $\alpha_{ij} = \alpha_{ji}$ such that $f(x, y) = \sum_{i,j=1}^{n} \alpha_{ij} x_i y_j$. (05)

b) Let $A, B \in M_n(F)$, show that $\det(AB) = \det A \cdot \det B$. (05)

c) Interchanging two rows of matrix changes the sign of its determinant. (04)

OR

Q-6 **Attempt all Questions** **(14)**

a) State and prove Cramer's rule. (05)

b) Identify the surface given by $11x^2 + 6xy + 19y^2 = 80$. Also convert it to the standard form by finding the orthogonal matrix P. (05)

c) Prove that determinant of lower triangular matrix is product of its entries on main diagonal. (04)